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LINEAR SUPERPOSITIONS WITH MAPPINGS WHICH LOWER DIMENSION

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ABSTRACT. It is shown that for every n-dimensional compact metric space X, there exist $2n+1$ functions $\{\varphi_j\}_{j=1}^{2n+1}$ in $C(X)$ and n mappings $\{\psi_i\}_{i=1}^{n}$ on X with 1-dimensional range each, with the following property: for every $0 \leqslant k \leqslant n$, every k tuple $\{\psi_{i_l}\}_{l=1}^{k}$ of the ψ_i's, and every $2(n-k)+1$ tuple $\{\varphi_{j_m}\}_{m=1}^{2(n-k)+1}$ of the φ_j's, each $f \in C(X)$ can be represented as $f(x) = \sum_{l=1}^{k} g_l(\psi_{i_l}(x)) + \sum_{m=1}^{2(n-k)+1} h_m(\varphi_{j_m}(x))$, with $g_l \in C(\psi_{i_l}(X))$ and $h_m \in C(R)$.

It is also shown that in many cases the number $2(n-k)+1$ is the smallest possible.

1. Introduction. Let X and Y_i, $1 \leqslant i \leqslant k$, be compact metric spaces, and let φ_i: $X \to Y_i$ be continuous functions. The family $\{\varphi_i\}_{1 \leqslant i \leqslant k}$ is said to be a measure separating family if each $f \in C(X)$ admits a representation

$$(*) \qquad f(x) = \sum_{i=1}^{k} g_i(\varphi_i(x))$$

with $g_i \in C(Y_i)$, $1 \leqslant i \leqslant k$.

$\{\varphi_i\}_{1 \leqslant i \leqslant k}$ is said to be a uniformly (point) separating family if each $f \in l_\infty(X)$ admits a representation $(*)$ with $g_i \in l_\infty(Y_i)$, $1 \leqslant i \leqslant k$. ($C(X)$ (resp. $l_\infty(X)$) is the Banach space of real continuous (resp. bounded) functions on X with the sup norm.)

As shown in **[9]** $\{\varphi_i\}_{1 \leqslant i \leqslant k}$ is measure separating (resp. uniformly separating) if and only if there exists a positive λ, such that for each $\mu \in M(X)$ (resp. $\mu \in l_1(X)$) there correspond some $1 \leqslant i \leqslant k$ so that

$$(**) \qquad \|\mu \circ \varphi_i\| \geqslant \lambda \|\mu\|.$$

(Here $M(X)$ is the Banach space of real Borel measures on X with the total variation as the norm; $l_1(X)$ is the subspace of $M(X)$ which consists of the purely atomic measures. Given an element μ of $M(X)$ (resp. $l_1(X)$) $\mu \circ \varphi_i$ is the element of $M(Y_i)$ (resp. $l_1(Y_i)$) defined by

$$\mu \circ \varphi_i(V) = \mu\left(\varphi_i^{-1}(V)\right)$$

where $V \subset Y_i$ is a Borel set.)

If the above is satisfied then $\{\varphi_i\}_{1 \leqslant i \leqslant k}$ is said to be a λ-measure separating family (resp. a uniformly separating family with constant λ). (This characterization motivated the choice of the terminology "measure separating" and "uniformly separating" family. See §§1 and 2 of **[9]** for a more detailed study of these concepts. Note

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also that by this characterization a measure separating family is also a uniformly separating family and hence if the linear superposition operator $(g_1, g_2, \ldots, g_k) \to \sum_{i=1}^{k} g_i \circ \varphi_i$ maps $C(Y_1) \times C(Y_2) \times \cdots \times C(Y_k)$ onto $C(X)$ then it also maps $l_\infty(Y_1) \times \cdots \times l_\infty(Y_k)$ onto $l_\infty(X)$. It is not clear whether the converse statement is valid. However, for $k = 1, 2$ it is. (See [**9**]).)

In this article we shall study the existence and nonexistence of measure separating and uniformly separating families in the case where the spaces Y_i, $1 \leqslant i \leqslant k$, are all of dimension smaller than the dimension of X. To begin with, let us mention two results, both related to the case where the functions φ_i are real valued. The first result is an extension of the well-known superposition theorem of Kolmogorov [**4**] and of a theorem of Ostrand [**8**]. We do not state this theorem in its most general version but in a form which suits the purpose of this article. For the general form as well as a proof, see [**10**].

THEOREM 1. *Let X be an n-dimensional compact metric space ($n \geqslant 0$). Then there exists a $1/(2n+1)$ measure separating family $\{\varphi_i\}_{i=1}^{2n+1} \subset C(X)$. Moreover, the set $\{(\varphi_1, \varphi_2, \ldots, \varphi_{2n+1}) \in C(X) \times C(X) \times \cdots \times C(X)$: $\{\varphi_i\}_{i=1}^{2n+1}$ is a $1/(2n+1)$ measure separating family$\}$ is residual in $C(X) \times \cdots \times C(X)$ ($2n+1$ factors), i.e. contains a dense G_δ set.*

The second theorem is an inverse of Theorem 1. By now it is known to be valid only for $n \leqslant 6$ [**9** and **12**], but I strongly believe in its validity for all n.

THEOREM 2. *Let X be an n-dimensional compact metric space ($n \geqslant 2$). Then no family which consists of $2n$ real-valued continuous functions on X is uniformly separating.*

Note that for $n = 1$ Theorem 2 is clearly false.

For a while I had the impression that Theorem 2 might be generalized in the following way: If X is an n-dimensional compact metric space, and if $\{\varphi_i\}_{i=}^{k}$ is a uniformly separating family so that $\dim \varphi_i(X) < n$ for $1 \leqslant i \leqslant k$, then $\sum_{i=1}^{k} \dim \varphi_i(X) \geqslant 2n+1$. However, after Professor A. Lelek brought to my attention his construction in [**6**], I was able to construct a measure separating family $\{\varphi_i\}_{i=1}^{4}$ on the square I^2 so that φ_i, $1 \leqslant i \leqslant 3$, are real valued, while φ_4 has a 1-dimensional range. This construction clearly refutes the above mentioned hypothesis. It turns out that this construction can be carried out in a more general setting which is our main existence theorem.

THEOREM 3. *Let X be an n-dimensional compact metric space ($n \geqslant 0$). Then there exist n continuous monotone functions $\{\psi_i\}_{i=1}^{n}$ on X, with 1-dimensional range each, and $2n+1$ functions $\{\varphi_j\}_{j=1}^{2n+1}$ in $C(X)$, so that for every $0 \leqslant k \leqslant n$ each k of the ψ_i's together with each $2(n-k)+1$ of the φ_j's forms a $\lambda_{n,k}$ measure separating family with $\lambda_{n,k} = 1/(2n+1+2k(n-k))$. Moreover, the functions ψ_i, $1 \leqslant i \leqslant n$, can be so chosen that the subset B of $[C(X)]^{2n+1}$ which consists of the elements $(\varphi_1 \cdots \varphi_{2n+1})$ which satisfy the above, is residual in $[C(X)]^{2n+1}$. In addition, if X is connected, locally-connected and unicoherent, then $\psi_i(X)$ is a dendrite, $1 \leqslant i \leqslant n$. (See §2 for definitions of these concepts.)*

We shall prove Theorem 3 in §3. In §4 we shall show that in many cases Theorem 3 is the best possible, i.e. that the numbers k and $2(n-k)+1$ cannot be reduced. This will follow from some slightly more general theorems.

Let us return to the false hypothesis mentioned above. By Theorem 3, with $k = n$, for each compact n-dimensional space X, there exist a measure separating family $\{\varphi_i\}_{i=1}^{n+1}$ with $\dim \varphi_i(X) = 1$ (more precisely $\dim \varphi_i(X) = 1$ for $1 \leqslant i \leqslant n$ and φ_{n+1} real valued), i.e. $\Sigma_i \dim \varphi_i(X) = n+1$, and if $n \geqslant 2$ then $\dim \varphi_i(X) < n$. But the functions φ_i, $1 \leqslant i \leqslant n$, are not real valued, i.e., the least Euclidean dimension into which $\varphi_i(X)$ can be embedded is $\geqslant 2$. So, if we define the *Euclidean index* $E(W)$ of a separable metric space W to be the smallest integer m such that W is homeomorphic to a subset of R^m (of course $\dim W \leqslant E(W) \leqslant 2 \dim W + 1$) then we obtain $\Sigma_{i=1}^{n+1} E(\varphi_i(X)) = \Sigma_{i=1}^{n} E(\varphi_i(X)) + 1 \geqslant 2n+1$. And in the general form of Theorem 1.3

$$\begin{aligned}\sum_{i=1}^{k} E(\psi_i(X)) + \sum_{j=1}^{2(n-k)+1} E\big(\varphi_j(X)\big) &= \sum_{i=1}^{k} E(\psi_i(X)) + 2(n-k) + 1 \\ &\geqslant 2k + 2(n-k) + 1 = 2n+1.\end{aligned}$$

These facts motivate the following problem, which trivially holds for $n \leqslant 1$, and, as will be shown in §4, holds also for $n = 2$ under some extra assumptions on the space X (e.g. if $\dim X = 2$ and X contains I^2).

Problem. Let X be an n-dimensional compact metric space, and let $\{\varphi_i\}_{i=1}^{k}$ be a uniformly separating family on X, with $\dim \varphi_i(X) < n$. Does it follow that $\Sigma_{i=1}^{k} E(\varphi_i(X)) \geqslant 2n+1$?

2. Preliminaries. In this section we mention some results and concepts that will be used in the following sections.

DEFINITION. (i) Let φ and ψ be mappings on a space X. ψ is said to be a refinement of φ if the cover $\{\psi^{-1}(y)\colon y \in \psi(X)\}$ of X is a refinement of the cover $\{\varphi^{-1}(z)\colon z \in \varphi(X)\}$.

(ii) Let F and G be two finite families of mappings on a space X. F is said to be a refinement of G if for each $\varphi \in G$ there correspond some $\psi \in F$ so that ψ refines φ.

The following proposition follows easily from the definitions, and its proof is left to the reader.

2.1. PROPOSITION. *Let X be a compact metric space and let F and G be finite families of mappings on X so that F refines G. If G is a measure (resp. uniformly) separating family then so if F.*

2.2. COROLLARY. *Let $F = \{\varphi_i\}_{i=1}^{k}$ be a measure (resp. uniformly) separating family on a compact metric space X. Let $a_1, a_2, \ldots, a_m$ be subsets of $\{1,2,\ldots,k\}$ so that $\bigcup_{j=1}^{m} a_j = \{1,2,\ldots,k\}$. Let $\psi_j\colon X \to \Pi_{i \in a_j} \varphi_i(X)$ be defined by $\psi_j(x) = (\psi_{i_1}(x), \psi_{i_2}(x), \ldots, \psi_{i_{|a_j|}}(x))$ where $a_j = \{i_1, i_2, \ldots, i_{|a_j|}\}$. Then $G = \{\psi_j\}_{j=1}^{m}$ is also a measure (resp. uniformly) separating family.*

2.3. COROLLARY. *Let* $F = \{\varphi_i\}_{i=1}^k$ *be as above. If for each* $1 \leqslant i \leqslant k$, φ_i *factors through some space* Y_i *with factors* ψ_i *and* τ_i, *i.e. there exist spaces* Y_i *and mappings* ψ_i: $X \to Y_i$ *and* τ_i: $Y_i \to \varphi_i(X)$ *so that* $\varphi_i(x) = \tau_i(\psi_i(x))$, *then* $H = \{\psi_i\}_{i=1}^k$ *is also a measure* (*resp. uniformly*) *separating family on* X.

PROOF OF COROLLARIES 2.2 AND 2.3. In both cases G and H refine F. □

Recall that a mapping φ: $X \to Y$ is called monotone if $\varphi^{-1}(K)$ is connected whenever $K \subset Y$ is closed and connected. The dimension of φ is defined by $\dim \varphi = \sup_{y \in Y} \dim \varphi^{-1}(y)$. The following are some results related to these concepts.

2.4. THEOREM (HUREWICZ). *If* φ: $X \to Y$ *is a closed mapping then* $\dim X \leqslant \dim Y + \dim \varphi$.

2.5. THEOREM (JUNG [2], KEESLING [3]). *If* φ: $X \to Y$ *is a closed mapping, and* $Y' = \{y \in Y\colon \dim \varphi^{-1}(y) \geqslant \dim X - \dim Y\}$ *then* $\dim X \leqslant \dim Y' + \dim \varphi$.

In both theorems the assumption on φ can be replaced by the assumption that X is σ compact.

2.6. THEOREM [5, p. 125]. *Let* X *be an n-dimensional space. Then there exists a* 0-*dimensional mapping* f: $X \to I^n$.

2.7. THEOREM [5, p. 184]. *Let* X *be a compact metric space, and* φ: $X \to Y$ *a mapping. Then* φ *factors through a space* T (*called the tree of* φ) *with factors* ψ: $X \to T$ *and* τ: $T \to Y$ *so that* ψ *is monotone*, $\dim \tau = 0$ *and* $\varphi(x) = \tau(\psi(x))$.

The elements of T can be identified with the components of the sets $\varphi^{-1}(y)$, $y \in Y$, $\psi(x)$ is then defined to be the component of $\varphi^{-1}(\varphi(x))$ which contains x, and $\tau(t) = \varphi(t) = y$ where t is a component of $\varphi^{-1}(y)$. As $\dim \tau = 0$ it follows from Theorem 2.4 that $\dim T \leqslant \dim Y$.

2.8. COROLLARY. *Let* $\{\varphi_i\}_{i=1}^k$ *be a measure* (*resp. uniformly*) *separating family on a compact space* X. *Then there exists a measure* (*resp. uniformly*) *separating family* $\{\psi_i\}_{i=1}^k$ *on* X *with* ψ_i *monotone and* $\dim \psi_i(X) \leqslant \dim \varphi_j(X)$, $1 \leqslant i \leqslant k$.

PROOF. Let ψ_i be the factor in the factorization of φ_i as in Theorem 2.7. Then ψ_i is monotone, $\dim \psi_i(X) \leqslant \dim \varphi_i(X)$, and by Corollary 2.3 $\{\psi_i\}_{i=1}^k$ is a measure (resp. uniformly) separating family. □

DEFINITION. (i) A compact metric space D which is connected, locally-connected and does not contain a homeomorphic image of the circle is called a dendrite.

(ii) A space X is called unicoherent if $X = X_1 \cup X_2$ with $X_i \subset X$, closed and connected, $i = 1, 2$, implies that $X_1 \cap X_2$ is connected.

The n-cube I^n is unicoherent for all $n \geqslant 1$. The proofs of the following facts about dendrites can be found in [5, p. 300].

2.9. A dendrite is unicoherent and 1-dimensional.

2.10. A dendrite can be embedded into R^2.

2.11. A compact connected subset of a dendrite is a dendrite.

2.12. A 1-dimensional connected and locally-connected compact unicoherent metric space is a dendrite.

2.13. Let X be a compact, connected, locally-connected and unicoherent metric space. If ψ is a monotone mapping on X with $\dim \psi(X) = 1$ then $\psi(X)$ is a dendrite.

DEFINITION. Let X be an n-dimensional space.

(i) A family F of n-dimensional closed subsets of X is said to be a *dimensional network for* X, if every n-dimensional closed subset of X contains some element of F.

(ii) X is said to be countably n-dimensional if it admits a countable dimensional network.

2.14. THEOREM [**11**]. *Let W be a compact subset of $Y_1 \times Y_2$ with* $\dim W = \dim Y_1 + \dim Y_2$. *If Y_i is countably* $\dim Y_i$*-dimensional for $i = 1$ or $i = 2$, then W contains a product $W \supset Y_1' \times Y_2'$ with* $\dim Y_i' = \dim Y_i$, $i = 1, 2$.

2.15. THEOREM. *A dendrite is countably* 1*-dimensional.*

PROOF. Let T be a dendrite and let E denote the set of end points of T, i.e. $E = \{t \in T\colon T \setminus \{t\}$ is connected$\}$. By a theorem of Menger [7] (see also [**1**, Theorem 4, p. 139]) $T \setminus E$ can be represented as a countable union of simple arcs $\{l_n\}_{n \geq 1}$ which do not intersect pair-wise in more than one point. (Note that E may be uncountable and may be dense in T. $T \setminus E$ is always dense in T.) For $n \geq 1$ let $\{l_{n,k}\}_{k \geq 1}$ be a sequence of simple arcs in l_n, whose interiors form a basis for the topology of l_n. We claim that $\{l_{n,k}\}_{n \geq 1, k \geq 1}$ is a dimensional network for T. Indeed, let $K \subset T$ be closed and 1-dimensional. Then K contains a 1-dimensional component M which by 2.11 is a dendrite too, and hence contains some arc l. Let $l' \subset l$ be another arc so that l' does not contain any of the two end points of l. Then each point of l' separates l' and hence separates T, i.e. $l' \subset T \setminus E = \bigcup_{n \geq 1} l_n$. Hence for some $n \geq 1$, $\dim(l' \cap l_n) = 1$ and thus $l' \cap l_n$ contains some arc, and also one of the arcs $l_{n,k}$ for some $k \geq 1$, and the theorem follows. $\square$

3. Proof of Theorem 3.

3.1. *The construction of Lelek.* In [**6**] A. Lelek constructs a mapping φ of I^n onto a dendrite T with some remarkable properties. In particular $D_\varphi = \{x \in I^n\colon \{x\} = \varphi^{-1}(\varphi(x))\}$ is a dense G_δ in I^n, and $E_\varphi = I^n \setminus D_\varphi$ has a simple explicit description. This mapping φ will play an important role in the proof of Theorem 3. We shall describe the construction below, giving detailed attention only to parts which will be relevant to our needs. The others can be found in [**6**]. Actually, we shall describe an upper semicontinuous decomposition L of I^n so that φ is the mapping induced by L, i.e. the elements of L are the fibers of φ.

Let Δ denote the classical Cantor set in [0, 1], and let P be an n-cube in R^n with boundary ∂P and center q. (At this point we are more restrictive than in the original construction [**6**]. There P could be any n-dimensional parallelepiped. The restriction to cubes will turn out to be more convenient later. Note that by an n-cube we mean any similar copy of the interior of I^n.) R^n will be looked upon as a vector space over R. Set $A = \{tx + (1-t)q\colon t \in \Delta \neq 1, x \in \partial P\}$.

A is a nowhere dense closed subset of P, $A \cup \partial P$ is compact and $\dim(A \cup \partial P) = n - 1$. $P \setminus A$ consists of countably many components, each of which is a domain D in R^n with boundary of the form $\partial D = \{t_i x + (1 - t_i)q\colon i = 1, 2, x \in \partial P\}$ where t_1

and t_2 are end points of some component interval of $[0, 1]\setminus\Delta$, i.e. $t_1 = k/3^m$ and $t_2 = (k+1)/3^m$ for some $m \geqslant 1$ and $1 \leqslant k \leqslant 3^m - 1$. Let us cut each of these domains D with finitely many compact pieces of $(n-1)$-dimensional hyperplanes contained in $\bar{D}$ to obtain finitely many congruent cubes P', so that each P' is of the form $P' = a + \alpha P$ where $a \in R^n$ and $0 < \alpha < 1$. (Actually if ∂D has the above representation with $t_1 = k/3^m$ and $t_2 = (k+1)/3^m$ then $\alpha = 1/3^m$. Note that here again we are more restrictive than in [**6**].) Let A' denote the union of A and of all these $(n-1)$-dimensional pieces, where D ranges over all the components of $P\setminus A$. (Note that there are countably many such pieces.) Let L_1 denote the family of components of A'. A_1 contains elements of two major types: components of the form $\{tx + (1-t)q\colon x \in \partial P\}$ with $1 \neq t \in \Delta$ and t not of the form $k/3^m$, and components of the form

$$\begin{aligned}\{k/3^m x + (1 - k/3^m)q\colon x \in \partial P\}\\ \cup\{((k+1)/3^m)x + (1-(k+1)/3^m)q\colon x \in \partial P\}\\ \cup\{\text{finitely many } (n-1)\text{-dimensional pieces}\}.\end{aligned}$$

Now we continue the construction as follows: The components of $P\setminus A'$ consists of countably many n-cubes of the form $a + \alpha P$. On each of these n-cubes we operate in the same way as we operated on P. In this way we obtain in each of these n-cubes sets similar to A and A'. The collection of components of $P\setminus\{$the union of all the A's (including the original A)$\}$ consists again of a countable family of n-cubes, and we continue by an obvious deduction. In this way we obtain countable collection of sets $\{A_l\}_{l=1}^{\infty}$ and $\{A'_l\}_{l=1}^{\infty}$ where for each $l \geqslant 1$, A'_l is of the form $A'_l = b_l + \beta_l A'$ ($b_l \in R^n$, $0 < \beta_l < 1$). Let us assume also that $A' = A'_1$. Let L_l denote the family of components of A'_l, and set

$$L = \bigcup_{l=1}^{\infty} L_l \cup \{\partial P\} \cup \left\{\text{the singletons of } P\setminus \bigcup_{l=1}^{\infty} A'_l\right\}.$$

Then L is a decomposition of $\bar{P}$. The reader may prove (or else check in [**6**]) that L is an upper semicontinuous decomposition of $\bar{P}$. Hence the quotient map φ induced by L maps $\bar{P}$ onto a compact metric space T. It is easy to check (see [**6**]) that $\dim T = 1$, and clearly the elements of L are connected, i.e., φ is a monotone mapping. Hence by 2.13 T is a dendrite. (Recall that $\bar{P}$ is homeomorphic to I^n and hence is locally-connected and unicoherent.) The sets $\overline{A'_l}$ are nowhere dense in $\bar{P}$, and hence, by the Bair category theorem $P\setminus \bigcup_{l=1}^{\infty} A_l$ is a dense G_δ. Since obviously $D_\varphi = \{x \in \bar{P}\colon \{x\} = \varphi^{-1}(\varphi(x))\} = P\setminus \bigcup_{l=1}^{\infty} A_l$, D_φ is a dense G_δ in P. Note also that $\bigcup_{l=1}^{\infty}\overline{A'_l} = \bigcup_{l=1}^{\infty} A'_l \cup \partial P$, and that $E_\varphi = \bar{P}\setminus D_\varphi = \bigcup_{l=1}^{\infty}\overline{A'_l}$. This completes the construction of φ.

We shall apply now the construction 3.1 to prove the following lemma.

3.2. LEMMA. *Let $X = I^n$ ($n \geqslant 2$). There exist n monotone mappings $\{\varphi_i\}_{i=1}^{n}$ on X, such that φ_i maps X onto a dendrite, and so that for each $1 \leqslant k \leqslant n$ and $1 \leqslant i_1 < i_2 < \cdots < i_k \leqslant n$, $\dim \bigcap_{l=1}^{k} E_{i_l} \leqslant n - k$, where $E_i = X\setminus D_i$ and $D_i = \{x \in X\colon \{x\} = \varphi_i^{-1}(\varphi_i(x))\}$. (Note that E_i is an F_σ in X.)*

Before giving the detailed proof, we shall describe intuitively the case $n = 2$ which exhibits the basic ideas. Let $P_1 = \{(x, y) \in R^2: |x| < 1, |y| < 1\}$ be the canonical square, and let $\varphi_1: \overline{P_1} \to T$ be the mapping associated to P_1 by the construction 3.1. Let P_2 be the square obtained from P_1 by rotation by 45° about the origin, and let $\varphi_2: \overline{P_2} \to T$ be the mapping associated to P_2 by 3.1. Set $X = \overline{P_1} \cap \overline{P_2}$. Obviously, X is homeomorphic to I^2, and we claim that the restriction to X of φ_1 and φ_2 satisfy the conditions of the lemma. For $k = 1$ this follows directly from 3.1. For $k = 2$ this is less obvious but still true: as mentioned in 3.1 $E_1 \subset \bigcup_{l=1}^{\infty} \overline{A'_{1,l}}$ where the $A'_{1,l}$ are the sets constructed in 3.1 with $P = P_1$. Respectively $E_2 \subset \bigcup_{l=1}^{\infty} \overline{A'_{2,l}}$. Hence, to prove $\dim(E_1 \cap E_2) \leqslant 0$ it suffices to show that $\dim(\overline{A'_{1,l_1}} \cap \overline{A'_{2,l_2}}) \leqslant 0$ for all l_1 and l_2. But this is intuitively clear: $\overline{A'_{1,l_1}}$ is the set constructed in 3.1, and $\overline{A'_{2,l_2}}$ is obtained from $\overline{A'_{1,l_1}}$ by rotation by 45°, and a possible translation and multiplication by a scalar. Thus the intersection does not contain any straight line segment, and hence is zero-dimensional.

The following proof uses the same ideas.

Proof of Lemma 3.2. Let $n \geqslant 2$, and let $\{e_{i,j}\}_{1 \leqslant i,j \leqslant n}$ be n^2 vectors in R^n so that:

(i) For $1 \leqslant i \leqslant n$, $\{e_{i,j}\}_{j=1}^{n}$ is an orthonormal basis for R^n.

(ii) $\dim \bigcap_{l=1}^{k} H_{i_l,j_l} = n - k$ for $1 \leqslant k \leqslant n$, $1 \leqslant i < i_2 < \cdots < i_k \leqslant n$, and $1 \leqslant j_l \leqslant n$ where $H_{i,j}$ is the subspace orthogonal to $e_{i,j}$.

For $1 \leqslant i \leqslant n$, let P_i be the canonical n-cube with center at the origin and volume 2^n whose sides are contained in the hyperplanes $\pm e_{i,j} + H_{i,j}$, $1 \leqslant j \leqslant n$. Set $X = \bigcap_{i=1}^{n} \overline{P_i}$. Then X is a convex body in R^n and hence homeomorphic to I^n. For $1 \leqslant i \leqslant n$, let φ_i be the mapping associated to P_i by the construction 3.1. We claim that the restrictions of the mappings φ_i to X satisfy the conditions of the lemma.

For $1 \leqslant i, j \leqslant n$ set $H'_{i,j} = e_{i,j} + H_{i,j} = \{e_{i,j} + x: x \in H_{i,j}\}$.

Then there exists a 0-dimensional F_σ subset V of $[-1, 1]$ so that $E_i \subset \bigcup_{j=1}^{n} V \cdot H'_{i,j}$, for all $1 \leqslant i \leqslant n$ (where $V \cdot H = \{t \cdot x: t \in V, x \in H\}$). To see this, fix some $1 \leqslant i \leqslant n$. In the following we shall omit the index i. Let A be the set defined in the construction 3.1. Then $\overline{A} = \Delta \cdot \partial P$. Now ∂P is contained in the union of the hyperplanes $\pm H'_j$, $1 \leqslant j \leqslant n$. So $\overline{A} \subset \bigcup_{j=1}^{n} (\Delta \cup (-\Delta)) \cdot H'_j$. The set $\overline{A'}$ has been obtained from $\overline{A}$ by adding to it countably many compact pieces of hyperplanes of the form $t \cdot H'_j$, $1 \leqslant j \leqslant n$, $-1 \leqslant t \leqslant 1$. Let τ denote the set of all those numbers t. Then

$$\overline{A'} \subset \bigcup_{j=1}^{n} (\Delta \cup (-\Delta) \cup \tau) \cdot H'_j = \bigcup_{j=1}^{n} V_1 \cdot H'_j,$$

where $V_1 = \Delta \cup (-\Delta) \cup \tau$. Clearly V_1 is a 0-dimensional F_σ in $[-1, 1]$. For a fixed $l \geqslant 1$, $\overline{A'_l} = b_l + \beta_l \overline{A'}$, $b_l \in R^n$, $0 \leqslant \beta_l \leqslant 1$. Let $b_l = (b_{l,1}, \ldots, b_{l,n})$ be the coordinates of b_l with respect to the basis $\{e_j\}_{j=1}^{n}$. Then

$$\overline{A'_l} = b_l + \beta_l \overline{A'} \subset b_l + \beta_l \bigcup_{j=1}^{n} V_1 \cdot H'_j = b_l + \bigcup_{j=1}^{n} \beta_l V_1 \cdot H'_j$$

$$= \bigcup_{j=1}^{n} (b_l + \beta_l V_1 \cdot H'_j) = \bigcup_{j=1}^{n} (b_{l,j} + \beta_l V_1) \cdot H'_j.$$

The last equality holds since $H_j' = H_j + e_j$ and H_j is the orthogonal subspace of e_j. Hence

$$\begin{aligned} b_l + \beta_l V_1 H_j' &= b_l + \beta_l V_1(e_j + H_j) = b_l + \beta_l V_1 e_j + H_j \\ &= (b_{l,j} + \beta_l V_1)e_j + b_l - b_{l,j}e_j + H_j \\ &= (b_{l,j} + \beta_l V_1)e_j + H_j \quad (\text{since } b_l - b_{l,j}e_j \in H_j) \\ &= (b_{l,j} + \beta_l V_1)(e_j + H_j) = (b_{l,j} + \beta_l V_1)\cdot H_j'. \end{aligned}$$

So, if we let $V_{l,j} = b_{l,j} + \beta_l V_1$, and $V_l = \bigcup_{j=1}^n V_{l,j}$ then $\overline{A_l'} \subset \bigcup_{j=1}^n V_{l,j}\cdot H_j' \subset \bigcup_{j=1}^n V_l \cdot H_j'$, and $V_{l,j}$ as well as V_l are 0-dimensional F_σ's in $[-1,1]$. Hence

$$E = \bigcup_{l=1}^{\infty} \overline{A_l'} \subset \bigcup_{l=1}^{\infty}\bigcup_{j=1}^{n} V_l\cdot H_j' \subset \bigcup_{j=1}^{n} V\cdot H_j'$$

where $V = \bigcup_{l=1}^{\infty} V_l$ is a 0-dimensional F_σ in $[-1,1]$.

We shall now prove the lemma with $\{i_1, i_2, \ldots, i_k\} = \{1, 2, \ldots, k\}$. By the above $\bigcap_{i=1}^k E_i \subset F = \bigcap_{i=1}^n \bigcup_{j=1}^n V\cdot H_{i,j}'$. Hence, it suffices to show that $\dim F \leqslant n-k$. But F can be also represented as $F = \bigcup \bigcap_{i=1}^k V\cdot H_{i,j}'$ where the union is taken over all the elements $(j_1, j_2, \ldots, j_k)$ of $\{1, 2, \ldots, n\}^k$, and since each of these united sets is σ compact it suffices to show that $\dim(\bigcap_{i=1}^k V\cdot H_{i,j_i}') \leqslant n-k$ for each $(j_1, j_2, \ldots, j_k)$. So fix $(j_1, \ldots, j_k)$. Define a mapping $f\colon \bigcap_{i=1}^k V\cdot H_{i,j_i}' \to V^k$ by $f(x) = (t_1, \ldots, t_k)$ if and only if $x \in \bigcap_{i=1}^k (t_i H_{i,j_i}')$. f is well defined since $(t_1, \ldots, t_k) \neq (t_1', \ldots, t_k')$ implies

$$\bigcap_{i=1}^{k} (t_i H_{i,j_i}') \cap \bigcap_{i=1}^{k} (t_i' H_{i,j_i}') = \varnothing.$$

(Actually $t_i \neq t_i'$ implies $t_i \cdot H_{i,j_i}' \cap t_i'\cdot H_{i,j_i}' = \varnothing$.) f is continuous, and by (ii)

$$\dim f^{-1}(t_1, \ldots, t_k) = \dim\left(\bigcap_{i=1}^{k} t_i\cdot H_{i,j_i}'\right) \leqslant n-k,$$

for all $(t_1, \ldots, t_k) \in V^k$, i.e. $\dim f \leqslant n-k$. As $\bigcap_{i=1}^k V\cdot H_{i,j_i}'$ is σ compact, it follows from Theorem 2.4 that

$$\dim \bigcap_{i=1}^{k} V\cdot H_{i,j_i}' \leqslant \dim V^k + \dim f \leqslant n-k$$

since $\dim V^k = 0$. This completes the proof of the lemma. $\square$

The next lemma is a generalization of Lemma 3.2 to general compact metric spaces.

3.3. LEMMA. *Let X be an n-dimensional compact metric space ($n \geqslant 2$). Then there exist monotone mappings $\psi_i\colon X \to S_i$, $1 \leqslant i \leqslant n$, so that* $\dim S_i = 1$, *and such that for each $1 \leqslant k \leqslant n$,* $\dim \bigcap_{l=1}^k F_{i_l} \leqslant n-k$, *where $F_i = X \setminus D_i$ and $D_i = \{x \in X\colon \{x\} = \psi_i^{-1}(\psi_i(x))\}$.*

PROOF. Let X be an n-dimensional compact metric space. Then (Theorem 2.6) X admits a 0-dimensional mapping $f\colon X \to I^n$. Let $\varphi_i\colon I^n \to T$ be the mappings guaranteed by Lemma 3.2. Let S_i be the tree of the mapping $\varphi_i \circ f\colon X \to T$ and let

ψ_i: $X \to S_i$ and τ_i: $S_i \to T$ be the mappings so that $\varphi_i \circ f = \tau_i \circ \psi_i$ as in Theorem 2.7. Recall that the elements of S_i are components of fibers of $\varphi_i \circ f$. $\psi_i(x)$ is the component which contains x, and $\tau_i(s) = \varphi_i(f(s))$, $s \in S_i$:

$$\begin{array}{ccc} X & \overset{\psi_i}{\to} & S_i \\ f\downarrow & \to & \downarrow\tau_i \\ I^n & \underset{\varphi_i}{\to} & T \end{array}$$

We claim that the $\{\psi_i\}_{i=1}^n$ do the job. Note first that by Theorem 2.7 dim $S_i = 1$, and the functions ψ_i are monotone. Also, $F_i \subset f^{-1}(E_i)$ where E_i are the subsets of I^n associated to φ_i in Lemma 3.2. Indeed, let $x \in F_i$. Hence there exists some $x' \neq x$ so that $\psi_i(x) = \psi_i(x') = s \in S_i$. Set $\tau_i(s) = t \in T$. Then s is a component of $(\varphi_i \circ f)^{-1}(t)$, because $(\varphi_i \circ f)(x) = \tau_i(\psi_i(x)) = \tau_i(s) = t$ and hence $x \in f^{-1}(\varphi_i^{-1}(t))$, and s is the component of the fiber of $\varphi_i \circ f$ which contains x. So, in particular $s \subset f^{-1}(\varphi_i^{-1}(t))$, and also $f(s) \subset \varphi_i^{-1}(t)$. But s is a compact connected set which contains more than one point $(x, x' \in s)$ hence $\dim s \geqslant 1$, and since $\dim f = 0$, $\dim f(s) \geqslant 1$ too, and in particular $f(s)$ contains more than one point. Thus, $f(s)$ is contained in $\varphi_i^{-1}(t)$, and is not a singleton, which means that $f(s) \subset E_i$, and $x \in S \subset f^{-1}(f(s)) \subset f^{-1}(E_i)$.

It follows that $\bigcap_{i=1}^k F_i \subset \bigcap_{i=1}^k f^{-1}(E_i) = f^{-1}(\bigcap_{i=1}^k E_i)$. But by Lemma 3.2 $\dim \bigcap_{i=1}^k E_i \leqslant n - k$, and since $\bigcap_{i=1}^k E_i$ is an F_σ and $\dim f = 0$, it follows from the Hurewicz Theorem 2.4 that $\dim f^{-1}(\bigcap_{i=1}^k E_i) \leqslant \dim \bigcap_{i=1}^k E_i + \dim f \leqslant n - k$, and the lemma follows. □

3.4. Lemma. *Let X be a compact metric space, and let $E \subset X$ be an F_σ with $\dim E \leqslant m$ $(m \geqslant 0)$. Then the subset A of $[C(X)]^{2m+1}$ defined by $A = \{(\varphi_1, \varphi_2, \ldots, \varphi_{2m+1}) \in [C(X)]^{2m+1}$: $\{\varphi_i\}_{i=1}^{2m+1}$ is a $1/(2m+1)$ measure separating family on $E\}$ is residual in $[C(X)]^{2m+1}$.*

Remark. The concept of a λ measure separating family has been defined on compact spaces. By a "λ measure separating family on E" we mean that for every $\mu \in M(X)$ so that the support of μ is contained in E, there correspond some φ_i so that $\|\mu \circ \varphi_i\| \geqslant \lambda \|\mu\|$.

Proof. Since E is an F_σ, it has a representation $E = \bigcup_{l=1}^\infty E_l$ with $E_l \subset E_{l+1}$, and E_l compact and $\dim E_l \leqslant m$. Set

$$A_l = \left\{(\varphi_1, \ldots, \varphi_{2m+1}) \in [C(X)]^{2m+1}: \right.$$
$$\left. \{\varphi_i\}_{i=1}^{2m+1} \text{ is a } \frac{1}{2m+1} \text{ measure separating family on } E_l \right\}$$

and

$$A_l' = \left\{(\psi_1, \ldots, \psi_{2m+1}) \in [C(E_l)]^{2m+1}: \right.$$
$$\left. \{\psi_i\}_{i=1}^{2m+1} \text{ is a } \frac{1}{2m+1} \text{ measure separating family on } E_l \right\}.$$

By Theorem 1, A'_l is residual in $[C(E_l)]^{2m+1}$, and hence $A'_l \supset \bigcap_{k=1}^{\infty} U'_k$ where U'_k are open and dense in $[C(E_l)]^{2m+1}$. Obviously, if $(\psi_1,\ldots,\psi_{2m+1}) \in A'_l$, and $(\varphi_1,\ldots,\varphi_{2m+1}) \in [C(X)]^{2m+1}$ is such that $\varphi_i/E_l = \psi_i$, $1 \leqslant i \leqslant 2m+1$, then $(\varphi_1,\ldots,\varphi_{2m+1}) \in A_l$. It follows that $A_l \supset \bigcap_{k=1}^{\infty} U_k = \{(\varphi_1,\ldots,\varphi_{2m+1})\colon (\varphi_{1/E_l},\ldots,\varphi_{2m+1/E_l}) \in U'_k\}$. We claim that U_k is open and dense in $[C(X)]^{2m+1}$. Let $\varphi = (\varphi_1,\ldots,\varphi_{2m+1}) \in U_k$. Then $\varphi/E_l = (\varphi_{1/E_l},\ldots,\varphi_{2m+1/E_l}) \in U'_k$. Hence, since U'_k is open, there exists some $\varepsilon > 0$ so that $\psi \in [C(E_l)]^{2m+1}$ and $|||\psi - \varphi_{/E_l}|||_{E_l} < \varepsilon$ implies $\psi \in U'_k$. (The norm in $[C(X)]^{2m+1}$ is $|||\varphi||| = \max_{1 \leqslant i < 2m+1} \|\varphi_i\|$, where $\|\cdot\|$ is the sup norm in $C(X)$.) It follows that if $\tau = (\tau_1,\ldots,\tau_{2m+1}) \in [C(X)]^{2m+1}$ and $|||\tau - \varphi|||_x < \varepsilon$, then in particular $|||\tau/_{E_l} - \varphi/_{E_l}|||_{E_l} < \varepsilon$ and hence $\tau/E_l \in U'_k$, i.e. $\tau \in U_k$, and thus U_k is open. To see that the U_k is dense, let $\tau \in [C(X)]^{2m+1}$ and $\varepsilon > 0$. U'_k is dense in $[C(E_l)]^{2m+1}$ hence there exists some $\psi \in U'_k$ so that $|||\tau/_{E_l} - \psi|||_{E_l} < \varepsilon$. Applying Tietze's extension theorem we can extend $\tau_i - \psi_i/E_l$ to an element $f_i \in C(X)$ so that $\|f_i\| < \varepsilon$, $1 \leqslant i \leqslant 2m+1$. Set $\hat{\psi}_i = f_i + \tau_i \in C(X)$. Then $\|\hat{\psi}_i - \tau_i\| = \|f_i\| < \varepsilon$ and $\hat{\psi}_i/E_l = \psi_i - \tau_i/E_l + \tau_i/E_l = \psi_i$, i.e. $\hat{\psi}_i$ is an extension of ψ_i, and hence $\hat{\psi} = (\hat{\psi}_1,\ldots,\hat{\psi}_{2m+1}) \in U_k$, and it follows that U_k is dense in $[C(X)]^{2m+1}$. Hence A_l is residual in $[C(X)]^{2m+1}$ since it contains $\bigcap_{k=1}^{\infty} U_k$, and $\bigcap_{l=1}^{\infty} A_l$ is residual too. We claim that $A = \bigcap_{l=1}^{\infty} A_l$. Clearly, if $\varphi \in A$ then $\varphi \in \bigcap_{l=1}^{\infty} A_l$. Let $\varphi \in \bigcap_{l=1}^{\infty} A_l$, and let μ be an element of $M(X)$ with $\|\mu\| = 1$ and $\operatorname{supp}\mu \subset E$. Let $\varepsilon > 0$. By the regularity of the measure μ, there exists some $l \geqslant 1$ so that $|\mu|(E_l) \geqslant 1 - \varepsilon$ (where $|\mu|$ is the variation of μ). Hence $\|\mu/E_l\| \geqslant 1 - \varepsilon$ and $\|\mu/_{X\setminus E_l}\| \leqslant \varepsilon$. $\varphi = (\varphi_1,\ldots,\varphi_{2m+1}) \in A_l$, hence, for some $1 \leqslant i \leqslant 2m+1$, $\|(\mu/E_l) \circ \varphi\| \geqslant 1/(2m+1)\cdot(1-\varepsilon)$ and since $\|\mu/_{X\setminus E_l}\| \leqslant \varepsilon$ we conclude that $\|\mu \circ \varphi_i\| \geqslant 1/(2m+1)(1-\varepsilon) - \varepsilon$, and since ε was arbitrary it follows that $\|\mu \circ \varphi_i\| \geqslant 1/(2m+1)$ for some $1 \leqslant i \leqslant 2m+1$, i.e. $\varphi \in A$. This proves the lemma. □

PROOF OF THEOREM 3. Let X be an n-dimensional compact metric space. The theorem is trivial if $n = 0$ (take φ to be an embedding of X into R) and also if $n = 1$. (Take $\{\varphi_j\}_{j=1}^{3} \subset C(X)$ a 1/3 measure separating family (by Theorem 1), and let ψ: $X \to X$ be the identity mapping.) So assume $n \geqslant 2$. Let $\{\psi_i\}_{i=1}^{n}$ be the mappings of Lemma 3.3. For $(i_1, i_2,\ldots,i_k) = \alpha$ where $1 \leqslant i_1 < i_2 < \cdots < i_k \leqslant n$, let $F_\alpha = F_{(i_1,\ldots,i_k)} = \bigcap_{j=1}^{k} F_{i_j}$ where F_i, $1 \leqslant i \leqslant n$, are the sets defined in Lemma 3.3. Recall that by this lemma F_α is an $(n-k)$-dimensional F_σ in X. Set

$$B_0 = \Big\{(\varphi_1, \varphi_2,\ldots,\varphi_{2n+1}) \in [C(X)]^{2n=1}\colon \{\varphi\}_{j=1}^{2n+1} \text{ is a } \frac{1}{2n+1} \text{ measure separating family on } X\Big\}$$

and for $\alpha = (i_1,\ldots,i_k)$ as above let A_α be the subset of $[C(X)]^{2(n-k)+1}$ defined by

$$A_\alpha = \Big\{(\varphi_1,\ldots,\varphi_{2(n-k)+1})\colon \{\varphi_j\}_{j=1}^{2(n-k)+1} \text{ is a } \frac{1}{2(n-k)+1} \text{ measure separating family on } F_\alpha\Big\}.$$

By Lemma 3.4 A_α is residual in $[C(X)]^{2(n-k)+1}$ and so is B_0 in $[C(X)]^{2n+1}$.

For $\beta = (j_1, j_2, \ldots, j_{2(n-k)+1})$ with $1 \leqslant j_1 < j_2 < \cdots < j_{2(n-k)+1} \leqslant 2n+1$, let $B_{\alpha,\beta}$ be the subset of $[C(X)]^{2n+1}$ defined by $B_{\alpha,\beta} = \{(\varphi_1, \ldots, \varphi_{2n+1})\colon \{\varphi_j\}_{j\in\beta}$ is in $A_\alpha\}$. (Note that $|\beta| = 2(n - |\alpha|) + 1$ where $|\cdot|$ stands for cardinality.)

Then $B_{\alpha,\beta}$ is residual in $[C(X)]^{2n+1}$ since $B_{\alpha,\beta}$ can be identified with the Cartesian product of $A_\alpha \subset [C(X)]^{2(n-k)+1}$ and $[C(X)]^{2k}$. It follows that

$$B_\alpha = \bigcap_\beta B_{\alpha,\beta} = \Big\{(\varphi_1, \ldots, \varphi_{2n+1}) \in [C(X)]^{2n+1}\colon \text{ for all } 1 \leqslant j_1 < \cdots < j_{2(n-k)+1} \leqslant 2n+1,\ \{\varphi_{j_l}\}_{l=1}^{2(n-k)+1} \text{ is a } \frac{1}{2(n-k)+1} \text{ measure separating family on } F_\alpha\Big\}$$

is residual in $[C(X)]^{2n+1}$, and thus the set $B = \bigcap_\alpha B_\alpha \cap B_0$ is residual in $[C(X)]^{2n+1}$ too.

We shall see now that each element $(\varphi_1, \ldots, \varphi_{2n+1})$ of B satisfies Theorem 3 (with respect to $\{\psi_i\}_{i=1}^n$ of course). Let k ψ's and $(2(n-k)+1)$ φ's be given. Without loss of generality we may assume that these are $\{\psi_i\}_{i=1}^k$ and $\{\varphi_j\}_{j=1}^{2(n-k)+1}$. Let $\alpha = (1, 2, \ldots, k)$. Then $(\varphi_1, \ldots, \varphi_{2n+1}) \in B_\alpha$ and hence $(\varphi_1, \ldots, \varphi_{2(n-k)+1}) \in A_\alpha$, i.e. $\{\varphi_j\}_{j=1}^{2(n-k)+1}$ is a $1/(2(n-k)+1)$ measure separating family on $F_\alpha = \bigcap_{i=1}^k F_i$. Recall also that $D_i = X \setminus F_i = \{x \in X\colon \{x\} = \psi_i^{-1}(\psi_i(x))\}$ and that $\psi_i(F_i) \cap \psi_i(D_i) = \varnothing$, $1 \leqslant i \leqslant n$. Let $\mu \in M(X)$ be of norm 1, and let δ be real with $0 \leqslant \delta \leqslant 1$. Consider the two possible cases

(i) $|\mu|(F_\alpha) < 1 - \delta$, and

(ii) $|\mu|(F_\alpha) \geqslant 1 - \delta$.

If (i) holds, then since $F_\alpha = \bigcap_{i=1}^k (X \setminus D_i) = X \setminus \bigcup_{i=1}^k D_i$, $|\mu|(\bigcup_{i=1}^k D_i) \geqslant \delta$ holds. Hence, $|\mu|(D_i) \geqslant \delta/k$ for some $1 \leqslant i \leqslant k$. ψ_i is one-to-one on D_i, and $\psi_i(D_i) \cap \psi_i(X \setminus D_i) = \varnothing$, and hence $\|\mu \circ \psi_i\| \geqslant \delta/k$.

If (ii) holds, then $\|\mu/F_\alpha\| \geqslant 1 - \delta$, and $\{\varphi_j\}_{j=1}^{2(n-k)+1}$ is a $1/r$ measure separating family on F_α (where $r = 2(n-k)+1$). Hence $\|\mu/F_\alpha \circ \varphi_j\| \geqslant (1-\delta) \cdot 1/r$ for some $1 \leqslant j \leqslant r$. Also, $|\mu|(X \setminus F_\alpha) < \delta$ and thus $\|\mu/_{X\setminus F_\alpha} \circ \varphi_j\| < \delta$ too. It follows that

$$\|\mu \circ \varphi_j\| \geqslant \|\mu/F_\alpha \circ \varphi_j\| - \|\mu/_{X\setminus F_\alpha} \circ \varphi_j\| > (1-\delta) \cdot 1/r - \delta = \frac{1}{r} - \frac{r+1}{r}\delta.$$

(Note that unlike case (i), in case (ii) we had to consider also the part of μ supported in $X \setminus F_\alpha$, since $\varphi_j(F_\alpha)$ and $\varphi_j(X \setminus F_\alpha)$ are not necessarily disjoint.) Thus in both cases $\|\mu \circ \tau\| \geqslant \min\{\delta/k,\ 1/r - (r+1)\delta/r\}$ for some $\tau \in \{\psi_i\}_{i=1}^k \cup \{\varphi_j\}_{j=1}^{2(n-k)+1}$. Hence if we take $\lambda_{n,k} = \sup_{0\leqslant\delta\leqslant 1} \min\{\delta/k,\ 1/r - (r+1)\delta/r\}$ then $\{\psi_i\}_{i=1}^k \cup \{\varphi_j\}_{j=1}^{2(n-k)+1}$ is a $\lambda_{n,k}$ measure separating family. The sup is clearly attained at δ which satisfies the equation $\delta/k = 1/r - (r+1)\delta/r$, i.e. $\delta = k/(r+k+rk)$ and $\lambda_{n,k} = 1/(r+k+rk) = 1/(2n+1+2k(n-k))$. Finally, from 2.13 it follows that if X is connected, locally-connected and unicoherent then $\psi_i(X)$ is a dendrite for $1 \leqslant i \leqslant n$. $\square$

4. Nonexistence theorems. In this section we shall show that in many cases Theorem 3 is the best possible. The precise statement is the following.

THEOREM 4. *Let X be an n-dimensional compact metric space* ($n \geqslant 2$), *and let* $\{\psi_i\}_{i=1}^k \cup \{\varphi_j\}_{j=1}^l$ *be a uniformly separating family on X with* $\{\varphi_j\}_{j=1}^l \subset C(X)$ *and* $\dim \psi_i(X) = 1$, $1 \leqslant i \leqslant k$, $0 \leqslant k \leqslant n$. *Then* $l \geqslant 2(n-k)+1$.

We shall prove this theorem in three different cases.

Case (i). $2 \leqslant n-k \leqslant 6$.

Case (ii). $n-k=1$.

Case (iii). $n-k=0$.

It turns out that the proof of each of these cases requires different methods. Moreover, in Case (iii) we shall need some additional assumptions on the space X. Note that the restriction $n-k \leqslant 6$ follows from the restriction $n \leqslant 6$ in Theorem 2. An extension of Theorem 2 for all $n \geqslant 2$ would imply the validity of Theorem 4 without the restriction $n-k \leqslant 6$. We shall not consider this problem in this article.

The following theorem implies Cases (i) and (ii) of Theorem 4.

THEOREM 5. *Let X be an n-dimensional compact metric space* ($n \geqslant 2$), *and let* $\{\varphi_j\}_{j=1}^l \cup \{\psi\}$ *be a uniformly separating family on X, with* $\{\varphi_j\}_{j=1}^l \subset C(X)$ *and* $\dim \psi(X) \leqslant k$, $1 \leqslant n-k \leqslant 6$. *Then* $l \geqslant 2(n-k)+1$.

Theorem 5 implies Cases (i) and (ii) of Theorem 4, since if $\{\varphi_j\}_{j=1}^l$ and $\{\psi_i\}_{i=1}^k$ are as in Theorem 4, then, by Corollary 2.2 $\{\varphi_j\}_{j=1}^l \cup \{\psi\}$ are as in Theorem 5, where ψ: $X \to \prod_{i=1}^k \psi_i(X)$ is defined by $\psi(x) = (\psi_1(x), \psi_2(x), \dots, \psi_k(x))$. ($\dim \prod_{i=1}^k \psi_i(X) = k$ since $\dim \psi_i(X) = 1$ for $1 \leqslant i \leqslant k$.)

PROOF OF THEOREM 5 (CASE (i)). Let $\{\varphi_j\}_{j=1}^l$ and ψ be as in Theorem 5 with $2 \leqslant n-k \leqslant 6$. Then by Theorem 2.4 there exists some $y \in \psi(X)$ with $\dim \psi^{-1}(y) \geqslant n-k$. Obviously $\{\varphi_j\}_{j=1}^l$ is a uniformly separating family on $\psi^{-1}(y)$. By Theorem 2, $l \geqslant 2(n-k)+1$. □

PROOF OF THEOREM 5 (CASE (ii)). We have to show that if $\varphi_1 \varphi_2$ and ψ are mappings on X with $\varphi_i \in C(X)$, $i=1,2$, and $\dim \psi(X) \leqslant n-1$, then $\{\varphi_1, \varphi_2, \psi\}$ is not a uniformly separating family. Assume that $\{\varphi_1, \varphi_2, \psi\}$ is a uniformly separating family on X. Without loss of generality we may assume that X is an n-dimensional Cantor manifold, and in particular that every open subset of X has dimension n. Then none of the families $\{\varphi_1, \psi\}$ and $\{\varphi_2, \psi\}$ is uniformly separating on any set $X' \subset X$ with nonempty interior in X. This follows from Theorem 6 (which will be proved independently later) since R is countably 1-dimensional. It follows that the mappings (φ_i, ψ): X: $\to R \times \psi(X)$ are 0-dimensional, $i=1,2$. Indeed, let $t_0 \in R$ and $y_0 \in \psi(X)$, and assume e.g. that $\dim(\varphi_1, \psi)^{-1}(t_0, y_0) \geqslant 1$. Then φ_2 is a homeomorphism on $\varphi_1^{-1}(t_0) \cap \psi^{-1}(y_0) = L$. Thus $\varphi_2(L)$ contains an open interval $J \subset R$, and $\varphi_2^{-1}(J) \setminus L$ is an open subset of X. Let $X' \subset \varphi_2^{-1}(J) \setminus L$ be compact with nonempty interior. We shall see that $\{\varphi_1, \psi\}$ is a uniformly separating family on X' which is a contradiction. Let $f \in l_\infty(X')$. Let $\hat{f} \in l_\infty(X)$ be an extension of f with $\hat{f}/L = 0$.

$\hat{f}$ admits a representation.

$\hat{f}(x) = g_1(\varphi_1(x)) + g_2(\varphi_2(x)) + g_3(\psi(x))$ with $g_i \in l_\infty(R)$, $i = 1,2$, and $g_3 \in l_\infty(\psi(X))$. We may assume without loss of generality that $g_1(t_0) = g_3(y_0) = 0$. Let

$x \in L$. Then $0 = \hat{f}(x) = g_1(\varphi_1(x)) + g_2(\varphi_2(x)) + g_3(\psi(x))$. But as $x \in L$, $\varphi_1(x) = t_0$ and $\psi(x) = y_0$, i.e.

$$0 = \hat{f}(x) = g_1(t_0) + g_2(\varphi_2(x)) + g_3(y_0) = g_2(\varphi_2(x)).$$

Hence g_2 vanishes on $\varphi_2(L)$. But $\varphi_2(X') \in J \subset \varphi_2(L)$, and thus φ_2 vanishes on $\varphi_2(X')$ too, i.e. for $x \in X'$ we have $f(x) = \hat{f}(x) = g_1(\varphi_1(x)) + g_3(\psi(x))$, which means that $\{\varphi_1, \psi\}$ is a uniformly separating family on X'.

From the fact that (φ_1, ψ) and (φ_2, ψ) are 0-dimensional, it follows that both φ_1 and φ_2 are 0-dimensional on $\psi^{-1}(y)$ for each $y \in \psi(X)$. ψ itself is a 1-dimensional mapping. ($\dim \psi = 0$ would imply $\dim \psi(X) \geqslant \dim X = n$, while $\dim \psi^{-1}(y) \geqslant 2$ for some $y \in \psi(X)$ is impossible by Theorem 2 since $\{\varphi_1, \varphi_2\}$ is uniformly separating on $\psi^{-1}(y)$.) Hence by Theorem 2.5 the set $Y = \{y \in \psi(X)\colon \dim \psi^{-1}(y) \geqslant 1\}$ has dimension $\geqslant n - 1$ and in particular $|Y| > \aleph_0$.

We shall construct now two disjoint sequences $\{x_j'\}_{j=1}^m$ and $\{x_j''\}_{j=1}^m$ of arbitrary length m in X, so that for each j, $\psi(x_j') = \psi(x_j'')$,

$$\varphi_1(x'_{2j-1}) = \varphi_1(x'_{2j}), \quad \varphi_1(x''_{2j-1}) = \varphi_1(x''_{2j}), \qquad j \geqslant 1,$$

and

$$\varphi_2(x'_{2j}) = \varphi_2(x'_{2j+1}), \quad \varphi_2(x''_{2j}) = \varphi_2(x''_{2j+1}), \qquad j \geqslant 1.$$

Once we do this, we can define $\mu \in l_1(X)$ by $\mu = \sum_{j=1}^m (-1)^j \delta_{x_j'} - \sum_{j=1}^m (-1)^j \delta_{x_j''}$ (where δ_x is the Dirac measure with mass 1 at x). Then $\|\mu\| = 2m$, $\mu \circ \psi = 0$, and $\|\mu \circ \varphi_i\| \leqslant 4$ for $i = 1, 2$, i.e. $\|\mu \circ \tau\| \leqslant 2/m \|\mu\|$ for all $\tau \in \{\varphi_1, \varphi_2, \psi\}$ which shows that $\{\varphi_1, \varphi_2, \psi\}$ is not a uniformly separating family.

Let B be a countable basis for the topology of R. For $J \in B$ let $Y_J = \{y \in Y\colon$ there exists a compact connected set $W_y \subset \psi^{-1}(y)$ so that $J \subset \varphi_1(W_y)\}$. Then $Y = \bigcup_{J \in B} Y_J$. Indeed, let $y \in Y$; then since $\dim \psi^{-1}(y) \geqslant 1$, $\psi^{-1}(y)$ contains some nontrivial connected compact set W_y, and since $\dim \varphi_1 = 0$ on $\psi^{-1}(y)$, $\dim \varphi_1(W_y) = 1$ and hence $\varphi_1(W_y)$ as a 1-dimensional subset of R contains some $J \in B$, i.e. $y \in Y_J$. It follows that for some $J \in B$, $|Y_J| > \aleph_0$. Note also that $J \times Y_J \subset (\varphi_1, \psi)(X)$. (If $t \in J$ and $y \in Y_J$, then $t \in J \subset \varphi_1(\psi^{-1}(y))$; hence for some $x \in \psi^{-1}(y)$, $\varphi_1(x) = t$ and thus $(\varphi_1, \psi)(x) = (t, y)$.) Let J' and J'' be two open intervals in J with $\bar{J}' \cap \bar{J}'' = \varnothing$, and set $X' = \varphi_1^{-1}(\bar{J}')$, $X'' = \varphi_1^{-1}(\bar{J}'')$. Then $X' \cap X'' = \varnothing$, $\dim X' = \dim X'' = n$ (since both have nonempty interior in X) and for all $y \in Y_J$, $\dim(X' \cap \psi^{-1}(y)) = \dim(X'' \cap \psi^{-1}(y)) = 1$. Indeed, let $y \in Y_J$. Then for some compact connected set $W_y \subset \psi^{-1}(y)$, $\varphi_1(W_y) \supset J \supset J'$ and hence $W_y \cap \varphi_1^{-1}(J') \subset X'$ is an open set relative to W_y; and since an open subset of a compact connected set is positive dimensional, $\dim(X' \cap \psi^{-1}(y)) \geqslant 1$. The same argument applies to X''.

Set $X_m' = X'$ and $Y_m' = Y_J$. We shall construct a decreasing sequence of n-dimensional compact sets $X_m' \supset X_{m-1}' \supset \cdots \supset X_1'$ and a decreasing sequence $Y_m' \supset Y_{m-1}' \supset \cdots \supset Y_1'$ of uncountable sets as follows: Applying the argument which has been used to show that $|Y_J| \geqslant \aleph_0$ for some $J \in B$, and the fact that $\dim(X_m' \cap \psi^{-1}(y)) \geqslant 1$ for all $y \in Y_m'$, we can show that for some open interval $I \subset \varphi_2(X_m') \subset R$ the set $Y_{m-1}' = \{y \in Y_m'\colon$ there exists a compact connected set

$W_y \subset \psi^{-1}(y) \cap X'_m$ so that $I \subset \varphi_2(W_y)\}$ is uncountable. Then $Y'_{m-1} \subset Y'_m$, $|Y'_{m-1}| > \aleph_0$ and $I \times Y'_{m-1} \subset (\varphi_2, \psi)(X'_m)$. Set $X'_{m-1} = \varphi_2^{-1}(\bar{I}) \cap X'_m$. Then $\dim X'_{m-1} = n$, and as before, for all $y \in Y'_m$, $\dim(X'_{m-1} \cap \psi^{-1}(y)) \geqslant 1$. Next, operate on X'_{m-1} in the very same way with φ_1 replacing φ_2, to obtain Y'_{m-2} and X'_{m-2}. Then operate on X'_{m-2} with φ_2 again to obtain Y'_{m-3} and X'_{m-3} and continue by an obvious induction till we end up with Y'_1 and X'_1.

Let us see now that for any m points $y_1, y_2, \ldots, y_m$ in Y'_1 with m even, there exist m points $\{x'_j\}_{j=1}^m$ in X'_m so that $\psi(x'_j) = y_j$, $\varphi_1(x'_{2j-1}) = \varphi_1(x'_{2j})$ and $\varphi_2(x'_{2j}) = \varphi_2(x'_{2j+1})$. So, let $\{y_j\}_{j=1}^m \subset Y'_1$. By the construction, there exists some interval $I \subset \varphi_1(X)$, so that $X'_1 = \varphi_1^{-1}(\bar{I})$ and $I \times Y'_1 \subset (\varphi_1, \psi)(X'_2)$. Let $x'_1 \in X'_1$ be such that $\psi(x'_1) = y_1$. (Such a point clearly exists. Actually $\dim(X'_1 \cap \psi^{-1}(y_1)) \geqslant 1$.) Set $t_1 = \varphi_1(x'_1)$. Then $t_1 \in I$, and y_1, y_2 are in Y'_1. Hence (t_1, y_1) and (t_1, y_2) are both in $I \times Y'_1 \subset (\varphi_1, \psi)(X'_2)$. It follows that there exists some $x'_2 \in X'_2$ with $(\varphi_1, \psi)(x'_2) = (t_1, y_2)$, i.e. $\psi(x'_2) = y_2$ and $\varphi_1(x'_2) = \varphi_1(x'_1) = t_1$. Let $s_2 = \varphi_2(x'_2)$. By the construction $X'_2 = \varphi_2^{-1}(\bar{J})$ where $J \subset R$ is some interval so that $J \times Y'_2 \subset (\varphi_2, \psi)(X'_3)$. Since $Y'_2 \supset Y'_1$ we can apply the same argument to find some $x'_3 \in X'_3$ with $\psi(x'_3) = y_3$ and $\varphi_2(x'_3) = \varphi_2(x'_2) = s_2$. Inductively we continue and construct $x'_4, \ldots, x'_m$ which satisfy our conditions. Let us return now to X''. Since $Y'_1 \subset Y_J = Y'_m$, $\dim(X'' \cap \psi^{-1}(y)) \geqslant 1$ for all $y \in Y'_1$. Hence we can repeat the same process with X'' replacing X' and Y'_1 replacing Y_J. In this way we obtain sets $Y''_m \supset Y''_{m-1} \supset \cdots \supset Y''_1$ with $|Y''_j| > \aleph_0$, $(Y''_m = Y'_1)$ and $X'' = X''_m \supset X''_{m-1} \supset \cdots \supset X''_1$. The same argument as above shows then that for any m points $\{y_j\}_{j=1}^m \subset Y''_1$ (m even) there exists m points $\{x''_j\}_{j=1}^m \subset X''_m = X''$ with $\psi(x''_j) = y_j$ as above. But since $\{y_j\}_{j=1}^m \subset Y''_1 \subset Y''_m = Y'_1$, we can also construct such points $\{x'_j\}_{j=1}^m$ with $\psi(x'_j) = y_j$ in $X'_m = X'$. Then $\{x'_j\}_{j=1}^m \cap \{x''_j\}_{j=1}^m = \varnothing$ (since $X' \cap X'' = \varnothing$) and we are done. This proves Case (ii) of Theorem 5 and also of Theorem 4. $\square$

The following theorem implies Case (iii) of Theorem 4, under some additional assumptions on the space X.

THEOREM 6. *Let X be an n-dimensional compact metric space ($n \geqslant 2$). Let Ψ_i: $X \to Y_i$ be mappings with* $\dim Y_i = n_i < n$, $i = 1, 2$, *and $n_1 + n_2 \leqslant n$. If Y_i is countably n_i-dimensional* (*see* §2) *for $i = 1$ or $i = 2$ then $\{\psi_1, \psi_2\}$ is not a uniformly separating family.*

PROOF. Assume that $\{\psi_1, \psi_2\}$ is a uniformly separating family. Then (ψ_1, ψ_2): $X \to Y_1 \times Y_2$ is an embedding, and hence $W = (\psi_1, \psi_2)$ is an n-dimensional compact subset of $Y_1 \times Y_2$. By Theorem 2.14 W contains a product $Y'_1 \times Y'_2 \subset W$ with $\dim Y'_i = n_i$, $i = 1, 2$. Let α_1, α_2 be points in Y'_1 and β_1, β_2 points in Y'_2. Then W contains the points (α_1, β_1), (α_1, β_2), (α_2, β_1) and (α_2, β_2). Let $\{x_j\}_{j=1}^4 \subset X$ be points so that

$$(\psi_1, \psi_2)(x_1) = (\alpha_1, \beta_1), \qquad (\psi_1, \psi_2)(x_2) = (\alpha_2, \beta_2),$$
$$(\psi_1, \psi_2)(x_3) = (\alpha_1, \beta_2) \quad \text{and} \quad (\psi_1, \psi_2)(x_4) = (\alpha_2, \beta_1)$$

and set $\mu = \delta_{x_1} + \delta_{x_2} - \delta_{x_3} - \delta_{x_4}$. Then $\mu \in l_1(X)$, $\|\mu\| = 4$ and $\mu \circ \psi_i = 0$ for $i = 1, 2$, which shows that $\{\psi_1, \psi_2\}$ is not a uniformly separating family. $\square$

PROOF OF THEOREM 4 (CASE (iii)) UNDER THE ASSUMPTION THAT X IS CONNECTED, LOCALLY-CONNECTED AND UNICOHERENT. Let X be as above, and let $\{\psi_i\}_{i=1}^n$ be mappings on X so that $\dim \psi_i(X) = 1$, $1 \leqslant i \leqslant n$. We wish to show that $\{\psi_i\}_{i=1}^n$ is not a uniformly separating family on X. If it is, then by Corollary 2.8 we may assume that the ψ_i's are monotone mappings. From 2.13 it follows then that $\psi_i(X)$ is a dendrite for $1 \leqslant i \leqslant n$, and by 2.15 $\psi_i(X)$ is countably 1-dimensional. Set $\psi = (\psi_2, \psi_3, \ldots, \psi_n)\colon X \to \prod_{i=2}^m \psi_i(X)$. Then $\dim \psi(X) \leqslant n - 1$, and by 2.2 $\{\psi_1, \psi\}$ is a uniformly separating family, contradicting Theorem 6. $\square$

COROLLARY. *Let X be a 2-dimensional compact, connected, locally-connected and unicoherent space, and let $\{\psi_i\}_{i=1}^m$ be a uniformly separating family on X, with $\dim \varphi_i(X) < \dim X$, $1 \leqslant i \leqslant m$. Then $\sum_{i=1}^m E(\varphi_i(X)) \geqslant 5$ where E is the Euclidean index defined in* §1.

PROOF. Assume that for k φ_i's, $E(\varphi_i(X)) \geqslant 2$, and for l φ_i's, $E(\varphi_i(X)) = 1$ (i.e. $\varphi_i \in C(X)$) where $k + l = m$. If $k = 0$ then by Theorem 4, Case (i), $l \geqslant 5$ and thus $\sum_{i=1}^m E(\varphi_i(X)) \geqslant 5$. If $k = 1$ then by Case (ii) of the same theorem $l \geqslant 2(2-1)+1 = 3$ and $\sum_{i=1}^m E(\varphi_i(X)) \geqslant 2 + 3 = 5$. If finally $k = 2$ then by Case (iii), $l \geqslant 1$ (here the special assumptions on X are applied) and again $\sum_{i=1}^m E(\varphi_i(X)) \geqslant 2 \cdot 2 + 1 = 5$. For $k \geqslant 3$ the corollary obviously holds. $\square$

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